16.7. Surface integrals : scalar functions

☆ **Def** Given a surface S parametrized by a vector function $\vec{r}(u,v)$ on a domain D, the surface integral of a scalar function f over S is

$$\iint_S f\,dS := \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

Note (1) Similar to line integrals of scalar functions, surface integrals of scalar functions do not have nice visual interpretations.

(2) The term $|\vec{r}_u \times \vec{r}_v|$ essentially serves as the Jacobian for the parameters u and v.

(3) Symmetry can be useful for surface integrals of scalar functions.

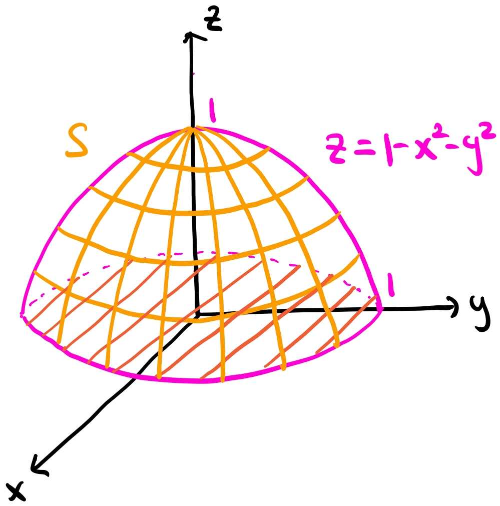
☆ **Prop** If a surface S is parametrized by $\vec{r}(u,v)$ on a domain D, then

$$\text{Area}(S) = \iint_S 1\,dS = \iint_D |\vec{r}_u \times \vec{r}_v|\,dA$$

Note The term $|\vec{r}_u \times \vec{r}_v|$ is reminiscent of the fact that the area of a parallelogram is given by the magnitude of the cross product.

Ex Find the area of the paraboloid $z=1-x^2-y^2$ with $z\geq 0$.

Sol 1 (Using the xy-parametrization)



The surface S is parametrized by

$\vec{r}(x,y) = (x, y, 1-x^2-y^2)$

The domain D is given by the shadow on the xy-plane

$\Rightarrow$ D is given by $x^2+y^2 \leq 1$.

$$\text{Area}(S) = \iint_S 1\,dS = \iint_D |\vec{r}_x \times \vec{r}_y|\,dA.$$

$$\vec{r}_x = (1, 0, -2x), \quad \vec{r}_y = (0, 1, -2y)$$

$$\Rightarrow \vec{r}_x \times \vec{r}_y = (2x, 2y, 1) \Rightarrow |\vec{r}_x \times \vec{r}_y| = \sqrt{4x^2+4y^2+1}$$

$$\Rightarrow \text{Area}(S) = \iint_D \sqrt{4x^2+4y^2+1}\,dA \quad (*)$$

In polar coordinates, D is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$.

Jacobian

$$\Rightarrow \text{Area}(S) = \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\cdot r\,dr\,d\theta$$

$$(u = 4r^2+1 \Rightarrow du = 8r\,dr)$$

$$= \int_0^{2\pi}\int_1^5 u^{\frac{1}{2}}\cdot\frac{1}{8}\,du\,d\theta = \int_0^{2\pi} \frac{u^{3/2}}{12}\Big|_{u=1}^{u=5} d\theta$$

$$= \frac{1}{12}\int_0^{2\pi} 5^{3/2}-1\,d\theta = \boxed{\frac{\pi}{6}(5^{3/2}-1)}$$

Note You can get the integral (*) using the surface area formula for graphs discussed in section 15.5.

Sol 2 (Using the cylindrical parametrization)

In cylindrical coordinates: $z = 1 - x^2 + y^2 \rightsquigarrow z = 1 - r^2$

The surface S is parametrized by

$\vec{S}(r, \theta) = (r\cos\theta,\ r\sin\theta,\ 1 - r^2)$

The domain D is given by $0 \le \theta \le 2\pi$, $0 \le r \le 1$.

$$\text{Area}(S) = \iint_S 1\, dS = \int_0^{2\pi} \int_0^1 |\vec{S}_r \times \vec{S}_\theta|\, dr\, d\theta.$$

$\vec{S_r} = (\cos\theta,\ \sin\theta,\ -2r)$, $\vec{S_\theta} = (-r\sin\theta,\ r\cos\theta,\ 0)$

$$\Rightarrow \vec{S_r} \times \vec{S_\theta} = (2r^2\cos\theta,\ 2r^2\sin\theta,\ r)$$

$$\Rightarrow |\vec{S_r} \times \vec{S_\theta}| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}.$$

$$\text{Area}(S) = \int_0^{2\pi} \int_0^1 r\sqrt{4r^2+1}\, dr\, d\theta = \boxed{\frac{\pi}{6}(5^{3/2} - 1)}$$

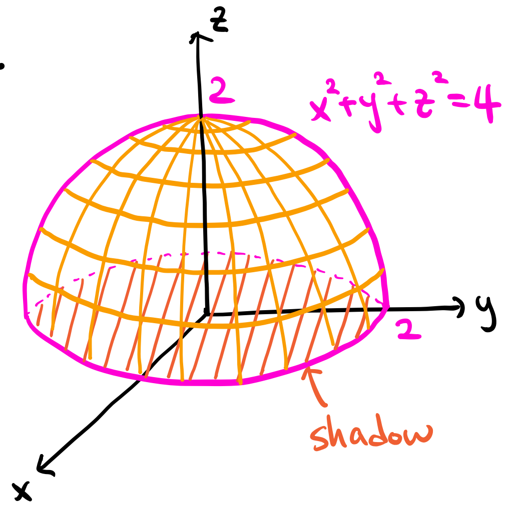
(same computation as in the first solution)

Note In the last integral, you should not multiply the Jacobian r for the cylindrical coordinate system because it is already taken care of by the term $|\vec{S_r} \times \vec{S_\theta}| = r\sqrt{4r^2+1}$

Ex Let S be the hemisphere $x^2+y^2+z^2=4$ with $z \geq 0$.

Find its center of mass with density $\rho(x,y,z)=1$.

Sol



$x^2+y^2+z^2=4 \rightsquigarrow z=\sqrt{4-x^2-y^2}$ $(z \geq 0)$

$\Rightarrow$ S is parametrized by

$\vec{r}(x,y) = (x, y, \sqrt{4-x^2-y^2})$.

The domain D is given by the shadow on the xy-plane

$\Rightarrow$ D is given by $x^2+y^2 \leq 4$.

$$m = \iint_S \rho(x,y,z)\,dS = \iint_S 1\,dS = \text{Area}(S) = \frac{1}{2}\cdot \underbrace{4\pi \cdot 2^2}_{\text{Area of sphere}} = 8\pi$$

S is symmetric about the xz, yz planes

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x,y,z)\,dS = \frac{1}{8\pi}\iint_S \underbrace{x}_{\text{odd w.r.t. } x}\,dS = 0.$$

$$\bar{y} = \frac{1}{m}\iint_S y\rho(x,y,z)\,dS = \frac{1}{8\pi}\iint_S \underbrace{y}_{\text{odd w.r.t. } y}\,dS = 0.$$

$$\bar{z} = \frac{1}{m}\iint_S z\rho(x,y,z)\,dS = \frac{1}{8\pi}\iint_S z\,dS$$

$$= \frac{1}{8\pi}\iint_D \sqrt{4-x^2-y^2}\;|\vec{r}_x \times \vec{r}_y|\,dA$$

$$\vec{r}_x = \left(1, 0, -\frac{x}{\sqrt{4-x^2-y^2}}\right), \quad \vec{r}_y = \left(0, 1, -\frac{y}{\sqrt{4-x^2-y^2}}\right)$$

$$\Rightarrow \vec{r}_x \times \vec{r}_y = \left(\frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1\right)$$

$$\Rightarrow |\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2} + 1} = \frac{2}{\sqrt{1-x^2-y^2}}.$$

$$\bar{z} = \frac{1}{8\pi}\iint_D \sqrt{1-x^2-y^2} \cdot \frac{2}{\sqrt{1-x^2-y^2}}\, dA = \frac{1}{8\pi}\iint_D 2\, dA$$

$$= \frac{1}{8\pi}\cdot 2\,\text{Area}(D) = \frac{1}{8\pi}\cdot 2 \cdot \underbrace{\pi \cdot 2^2}_{\text{Area of disk}} = 1$$

$\Rightarrow$ The center of mass is $\boxed{(0, 0, 1)}$

Note As discussed in Lecture 35, you can use other parametrizations :

$$\begin{cases} \vec{s}(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{4-r^2}) \text{ with } 0 \le \theta \le 2\pi,\ 0 \le r \le 2 \\ \vec{t}(\theta, \varphi) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi) \\ \qquad \text{with } 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$

However, it is not easy to use symmetry with these parametrizations.